J. Appl. Maths Mechs Vol. 56, No. 1, pp. 102–107, 1992
Printed in Great Britain.

0021–8928/92 $15.00+.00

ON THE DOMAIN OF VARIATION OF ADDED MASSES, POLARIZATION AND EFFECTIVE CHARACTERISTICS OF COMPOSITES†

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(*Received* 13 *November* 1990)

The well-known result of Pólya and Schiffer concerned with an estimate for the added mass (AM) and polarization tensors is improved. The exact domain of variation of the AM and polarization tensors for a body of fixed volume is found. It is shown that the estimates of the possible values of the effective conductivity matrix known in the theory of composite materials are direct analogues of the corresponding estimates for the AM and polarization. The proposed method of proof differs only by longer calculations in the theory of composites, but opens a wider perspective for studying a number of other problems in the theory of composites. Moreover, the exact value of the AM is calculated for two independently moving cylinders at the moment of contact.

IN IDEAL incompressible fluid dynamics the Pólya–Schiffer inequality [1] is known to hold for added mass (AM). According to this inequality, the AM of a body averaged over the directions is not less than that of a sphere of the same volume. Strangely enough, no answer to the question concerning the domain of possible values of the AM tensor for a body of a given volume can be found in the literature.

At the same time, in the case of the more complex problem of estimating the effective conductivity of a two-phase composite, an answer has been obtained and constitutes the contents of the well-known Hashin–Shtrickman–Lurie–Cherkayev–Murat–Tartar two-sided estimates X–T [2–4]. As has already been pointed out in [5], the Pólya–Schiffer estimate is the low-temperature limit of the X–T two-sided estimate. As a result of a more careful examination of the original proof of Pólya and Schiffer, we discovered that, firstly, it enables one to give a complete answer to the question concerning the domain of variation of the AM, and, secondly, after a slight modification, it can be carried over to the case of two-phase composites, yielding a direct proof of the X–T estimate. Let us note that one of the Hashin–Shtrickman estimates has been obtained earlier in the same way in [6], where the analogy has also been pointed out. Although many proofs of the X–T two-sided estimate are known at present (see [7–9]), they are not so direct and employ deeper concepts.

We became interested in the study of the AM in connection with the problem of determining the force of impact of a drifting iceberg hitting a drilling platform (for an experimental and numerical discussion of the problem, see e.g. [10]). Here we are concerned with evaluating the AM of a cylinder when another stationary cylinder is present at the moment of contact, rather than for a cylinder immersed in an unbounded liquid. Below we shall give a simple exact solution of this problem using a conformal mapping.

1. DOMAINS OF VARIATION OF ADDED MASSES AND POLARIZATION

We recall that by the added mass (AM) tensor $m = (m_{ij})$ we mean the value of the following variational problem, in which the density of the fluid is assumed to be equal to unity:

† *Prikl. Mat. Mekh.* Vol. 56, No. 1, pp. 118–123, 1992.

$$m\xi\cdot\xi = \sup_{\Phi}\left\{2\int_{\partial V}\Phi n\cdot\xi\, dS - \int_{V}|\nabla\Phi|^2 dV\right\}$$

Similarly, the value of the problem

$$b\xi\cdot\xi = \sup_{\Phi}\left\{2\int_{\partial V}\frac{\partial\Phi}{\partial n} y\cdot\xi\, dS - \int_{V}|\nabla\Phi^2|\, dV\right\}$$

is called the polarization tensor $b = (b_{ij})$.

The following theorem answers the question concerned with the AM (the polarization) of a body of fixed volume v.

Theorem 1. The eigenvalues $M_j(B_j)$ of the AM (polarization) tensor for a body of volume v lie in the domain determined by the inequalities

$$M_j > 0, \quad j = 1, \ldots, n; \quad \sum_{j=1}^{n}\frac{1}{v+M_j} \leqslant \frac{n-1}{v} \tag{1.1}$$

$$B_j > 0, \quad j = 1, \ldots, n; \quad \sum_{j=1}^{n}\frac{1}{v+B_j} \leqslant \frac{1}{v} \tag{1.2}$$

The proof of (1.1) consists in substituting the simple layer potential

$$\Phi = \int_{\partial V}\zeta\cdot nG(x-y)\, dS_y \tag{1.3}$$

into the variational representation for the AM. Here G is the fundamental solution of the Laplace operator ($\Delta G = \delta$). On using Green's formula and the jump of the normal derivative of the simple layer in the resulting integral over V, we apply the Cauchy inequality to estimate the Dirichlet integral. Then, after making the optimal choice of $\zeta \in R^n$, we obtain the matrix inequality

$$(M^{-1}+v^{-1})^{-1} \geqslant \Pi, \quad \Pi = \{\pi_{ij}\} = \left\{\int_V\int_V \frac{\partial^2 G(x-y)}{\partial x_i \partial x_j}\, dx\, dy\right\} \tag{1.4}$$

It should be mentioned that $\mathrm{Tr}\,\Pi = v$, as follows from (1.4) and the equality $\Delta G = \delta$. To obtain the inequalities (1.2), one should proceed in the same way using the double-layer potential

$$\Phi_* = \int_{\partial V}\zeta\cdot y\, \frac{\partial G(x-y)}{\partial n_y}\, dS_y \tag{1.5}$$

As can be seen from the proof presented above, inequalities (1.1) are exact if the harmonic function inside V, whose boundary value is equal to that of the AM potential on ∂V, is linear in V. It is known that this is the case for ellipsoids. To construct a body of a given volume with the AM from the domain determined by inequalities (1.1), it suffices to attach suitable flat pieces to such an ellipsoid so that the volume remains unchanged but the AM is increased in the appropriate direction.

It is interesting to note that $\zeta = \xi$ in Schiffer's proof. This choice is exact only for a sphere and suffices for the correct estimate of the trace of the AM only. The Pólya–Schiffer estimate can be obtained from (1.1) if, in addition, one considers the inequality between the arithmetic mean and the geometric mean.

We will also mention a useful inequality connecting the AM M_1 of a body B_1 with the AM M_2 of a body B_2 contained in B_1. If we denote their volumes by v_1 and v_2, then the tensor inequality $M^1 + v_1 \geqslant M^2 + v_2$ will be satisfied. The inequality can be obtained by substituting the AM potential of B_2 as a test function into the variational problem that yields M_1. It is, however, more convenient to understand this inequality in the context of the theory of composites by considering two composites, one with the periodic inclusion B_1 and the other one with B_2, and assuming that these are non-conducting inclusions. Then, of course, the first composite has smaller effective conductivity, which yields the desired inequality in the low-concentration limit [5].

2. THE X–T TWO-SIDED ESTIMATE

Let $0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \infty$ and let ν, where $0<\nu<1$, be the part of the volume occupied by the phase σ_1. The effective conductivity $\sigma^\circ = (\sigma_{ij}^\circ)$ of the composite inside the cube $[0, 1]^n$ can be determined from the relation

$$\sigma_{ij}^\circ \xi_i \xi_j = \inf_{\langle \nabla \Phi \rangle = \xi} \int_{[0,1]^n} \sigma(x)(\nabla\Phi)^2 \, dx \tag{2.1}$$

$$\overset{\circ}{\sigma}{}_{ij}^{-1} \xi_i \xi_j = \inf_{\substack{\langle p \rangle = \xi \\ \partial_i p_i = 0}} \int_{[0,1]^n} \sigma^{-1}(x) |p|^2 \, dx \tag{2.2}$$

$$(\sigma(x) = \sigma_1 \chi(x) + \sigma_2(1 - \chi(x)), \quad \langle \chi \rangle = \nu)$$

where χ is the characteristic function of V, $\sigma^{\circ -1}$ is the inverse matrix to σ° and the minimum is sought in the set of periodic functions in (2.1) and vector-valued functions in (2.2), $\langle \cdot \rangle$ being the average of a periodic function.

According to the X–T two-sided estimate, the eigenvalues $\{\lambda_j\}$ of σ° occupy the domain defined by the inequalities

$$(-1)^{\alpha-1} \sum_{j=1}^{n} \frac{1}{\lambda_j - \sigma_\alpha} \leqslant \frac{1}{\sigma_\Gamma - \sigma_\alpha} + \frac{n-1}{\sigma_a - \sigma_\alpha}, \quad \alpha = 1, 2 \tag{2.3}$$

$$\sigma_\Gamma \leqslant \lambda_j \leqslant \sigma_a \quad j = 1, \ldots, n$$

$$(\sigma_a = (\sigma_1 \nu + \sigma_2(1 - \nu)), \quad \sigma_\Gamma = (\sigma_1^{-1}\nu + \sigma_2^{-1}(1 - \nu))^{-1})$$

To derive the estimate (2.3) for $\alpha = 2$, we use the representation (2.1) and, by analogy with (1.4), we take a test function of the form

$$\Phi(x) = x \cdot \xi + \psi(x), \ \Psi(x) = \int_{\partial V} G(x - y) \, n \cdot \zeta \, dS_y \xi, \ \zeta \in R^n \tag{2.4}$$

where $G(z)$ is the periodic fundamental solution of the Laplace operator:

$$\Delta_z G(z) = \sum_{k \in Z^n} \delta(z - k) - 1 \tag{2.5}$$

Here Z^n is the integer-valued lattice. It is necessary to subtract one in order to obtain a periodic function G.

A calculation using formula (1.5) to write down the energy in the domain V occupied by σ_1 leads to the relation

$$(\sigma^\circ - \sigma_a)\xi\cdot\xi \leqslant \int_V [2(\sigma_1 - \sigma_2)\cdot \xi \cdot \nabla\psi + \sigma_2 \zeta \cdot \nabla\psi] \, dx + (\sigma_1 - \sigma_2) \int_V |\nabla\psi|^2 \, dx$$

Estimating the last term by means of the Caucy inequality

$$\int_V |\nabla\psi|^2 \, dx \geqslant \frac{1}{\nu} \left| \int_V \nabla\psi \, dx \right|^2$$

and setting $\Pi\zeta = \int_V \psi \, dx$, on choosing the optimal ζ, we arrive at the matrix inequality

$$\frac{\sigma^\circ - \sigma_a}{(\sigma_1 - \sigma_2)^2} \leqslant -\left(\sigma_2 \Pi^{-1} + \frac{\sigma_1 - \sigma_2}{V}\right)^{-1}$$

Solving this inequality for Π and using the relation $\mathrm{Tr}\,\Pi = \nu(1-\nu)$, we obtain inequality (2.3) for $\alpha = 2$.

To derive this inequality for $\alpha = 1$, which turns out to be more difficult from the computational point of view, one should take a test function of the form

$$p = \nabla\Phi_*, \quad \Phi_* = x \cdot \xi + \psi_*, \quad \psi_*(x) = \int_{\partial V_*} y \cdot \zeta \, \frac{\partial G(x - y)}{\partial n} \, dS_y$$

$$V_* = I^n \setminus V, \ I^n = [0, 1]^n$$

in (2.2). Such a vector-valued function is admissible, since the normal derivative of a double layer is continuous.

Using Green's formula for the jump of the double layer potential, we obtain

$$\int \sigma_* |p|^2 dx = \sigma_a{}^* |\xi|^2 + 2\int_{I^n} \sigma_* \xi \cdot \nabla\psi \, dx + \int_{I^n} \sigma_* |\nabla\psi|^2 dx$$

$$\sigma_* = \sigma^{-1}(x),\ \sigma_1{}^* = \sigma_2{}^{-1},\ \sigma_2{}^* = \sigma_1{}^{-1},\ \sigma_a{}^* = \sigma_1{}^* v_* + \sigma_2{}^* v$$

$$\int_{I^n} \sigma \xi \cdot \nabla\psi \, dx = \sigma_1{}^* K_* + \sigma_2{}^* K = (\sigma_2{}^* - \sigma_2{}^*) K_* + \sigma_2{}^* v_* \zeta \cdot \xi$$

$$K = \int_V \xi \cdot \nabla\psi \, dx, \quad K_* = \int_{V_*} \xi \cdot \nabla\psi \, dx$$

Here we have used the transformation

$$K = \int_{\partial V} n \cdot \xi \psi \, dS, \quad K_* = \int_{\partial V_*} \psi_{in} n \cdot \xi \, dS + \int_{\partial V_*} \zeta \cdot y \cdot n \cdot \xi \, dS$$

where ψ_{in} is the limiting value of ψ from within the domain. By analogy, we have

$$\int_{V_*} \sigma_* |\nabla\psi|^2 dx = (\sigma_1{}^* - \sigma_2{}^*) L_* + \sigma_2{}^* \int \zeta \cdot \nabla\psi \, dS, \quad L_* = \int_{V_*} |\nabla\psi|^2 dx$$

Besides $\langle \nabla\psi \rangle = v_* \zeta$, and so, replacing ξ by $\xi + v_* \zeta$, we get

$$\sigma^{\circ *}(\xi + v_* \zeta) \cdot (\xi + v_* \zeta) \leqslant \langle \sigma^{-1} |p|^2 \rangle = 2(\sigma_1{}^* - \sigma_2{}^*)\, \xi \cdot \int_{V_*} \nabla\psi \, dx + \sigma_2{}^* \zeta \cdot \int_{V^*} \nabla\psi \, dx + $$
$$+ 2\sigma_2{}^* v \xi \cdot \zeta + \sigma_a{}^* |\xi|^2 + L_*$$

where $\sigma^{\circ *}$ is the inverse matrix to σ°. The last term can be estimated by means of the Cauchy inequality, as a result of which we arrive at the inequality

$$(\sigma^{\circ *} \xi, \xi) \leqslant 2\xi \cdot A\zeta + B\zeta \cdot \zeta + \sigma_a{}^* |\xi|^2$$

$$A = (\sigma_1{}^* - \sigma_2{}^*) \cdot (\Pi_* - v_*), \quad B = \frac{\sigma_1{}^* - \sigma_2{}^*}{v_*} \cdot \Pi_*{}^2 - 2(\sigma_1{}^* - \sigma_2{}^* v_*) \cdot \Pi_* +$$
$$+ \sigma_2{}^* \cdot \Pi_* + (\sigma_a{}^* - 2\sigma_2{}^*) v_*{}^2, \quad \Pi_* \zeta = \int_{V_*} \nabla\psi_* \, dx$$

It is remarkable that B is divisible by A as a polynomial in Π_*. Thus, by optimizing ζ, we can obtain the matrix inequality

$$\sigma^{\circ *} - \sigma_a{}^{\circ *} \leqslant -AB^{-1}A = -v_* (\Pi_* - \alpha \cdot v_*)^{-1} (\sigma_1{}^* - \sigma_2{}^*)(\Pi_* - v_*{}^2)$$

where $\alpha = (\sigma_a{}^* - 2\sigma_2{}^*)(\sigma_1{}^* - \sigma_2{}^*)^{-1}$.

Solving this inequality for Π_* and using the relation $\mathrm{Tr}\,\Pi_* = n \cdot v_* + v_*(1 - v_*)$, we arrive at the inequality (2.3) for $\alpha = 1$. The above estimates turn into equalities in the case when the extremal field is linear on the phase $\sigma_1{}^*$ (in a medium formed by attired ellipsoids [2]).

For $\alpha = 2.1$, the estimates (1.1) and (1.2) can be obtained from (2.3) with $\sigma_1 = 0$, $\sigma_2 = 1$ and $\sigma_1 = 1$, $\sigma_2 = \infty$, respectively, by passing to the small concentration limit for the periodic inclusion. This passage to the limit preserves the second or the first terms on the right-hand side of (2.3), respectively. This assertion can be obtained by the method described in [5].

3. THE ADDED MASS OF A CYLINDER IN THE PRESENCE OF A STATIONARY CYLINDER

There are two circles: S_R of radius R and S_r of radius r. The circles are tangent to one another, forming a figure-of-eight. We consider the problem of finding the harmonic function Φ inside the figure-of-eight with boundary conditions

$$\partial\Phi/\partial n\ |_{S_R} = 0, \quad \partial\Phi/\partial n\ |_{S_r} = n\cdot\xi$$

n being the outer normal vector to S_r. It is required that the AM tensor

$$m\xi\cdot\xi = \int_{S_r} \Phi \frac{\partial\Phi}{\partial n} dS = \int_{S_r} \Phi n\cdot\xi\, dS$$

be evaluated.

Theorem 2. The tensor (m_{ij}) is spherical, $m_{ij} = m\cdot\delta_{ij}$ with $m = m(r, R)$, where

$$m(r, R) = \pi r^2\Bigl(1 + 2\sum_{n=1}^{\infty}\frac{1}{(2\beta n+1)^2}\Bigr), \quad 2\beta = I + \frac{r}{R} \tag{3.1}$$

In particular,

$$m = \pi r^2\,(\pi^2/3 - 1)$$

for $R = \infty$.

To obtain (3.1), we must apply inversion with respect to the point of contact. Then we have to evaluate the Dirichlet integral of the harmonic function in the strip $-1/(2r) \leqslant x_1 \leqslant 1/(2R)$ that satisfies the conditions

$$\frac{\partial u}{\partial x_1}\Bigr|_{x_1=1/(2R)} = 0, \quad \frac{\partial}{\partial x_1}\Bigl(u - \frac{x_1\xi_1 - x_2\xi_2}{x_1^2 + x_2^2}\Bigr)\Bigr|_{x_1=-1/(2r)} = 0$$

On applying a Fourier transformation $(x_2 \to k)$ and solving the ordinary differential equation in x_1, we obtain

$$u(x_1, k) = \frac{f^*(k)}{k}\Bigl[\operatorname{sh}\Bigl(k\frac{R+r}{2Rr}\Bigr)\Bigr]^{-1}\operatorname{ch}\Bigl(k\Bigl(x_1 + \frac{1}{2R}\Bigr)\Bigr)$$

$$f^*(k) = F_{x_2\to k}\Bigl(\frac{(x_2^2 - {}^1\!/_4 r^{-2})}{(x_2^2 + {}^1\!/_4 r^{-2})}\Bigr) = \sqrt{\frac{\pi}{2}}\,k\exp\Bigl(-\frac{k}{2r}\Bigr)$$

for $\xi = (1, 0)$.

Evaluating the Dirichlet integral in terms of x_1 and k using the Parseval equality, we obtain

$$m(r, R) = \pi\int_0^{\infty} k\operatorname{cth}\Bigl(k\frac{R+r}{2Rr}\Bigr)\exp\Bigl(-\frac{k}{r}\Bigr)dx$$

Hence, expanding the cth function in a geometric progression, we obtain (3.1). The series in (3.1) can be expressed in terms of the gamma-function [11]:

$$m(r, R) = \pi r^2\Bigl(1 + \frac{1}{2\beta^2}\frac{d^2}{dz^2}\ln\Gamma(z)\,|_{z=1/(2\beta)}\Bigr)$$

In a similar way we can evaluate the coefficient of proportionality between the force acting on the stationary cylinder and the acceleration of the moving cylinder:

$$m_{12} = \int dx_2 \int_{-{}^1\!/_2 R^{-1}}^{{}^1\!/_2 r^{-1}} dx_1 \nabla u\cdot\nabla u^*$$

where u^* is harmonic in the strip and satisfies the following boundary conditions:

$$\frac{\partial u^*}{\partial x_1}\Bigr|_{x_1={}^1\!/_2 r^{-1}} = 0 \quad \frac{\partial}{\partial x_1}\Bigl(u^* - \frac{x_1}{x_1^2 + x_2^2}\Bigr)\Bigr|_{x_1=-{}^1\!/_2 R^{-1}} = 0$$

A similar argument yields

$$m_{12} = \pi R_h^2 \cdot \pi^2/12 \quad (R_h = ({}^1/_2 R^{-1} + {}^1/_2 r^{-1})^{-1})$$

I am grateful to G. S. Kulikov for drawing this problem to my attention.

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Translated by T.Z.

J. Appl. Maths Mechs Vol. 56, No. 1, pp. 107–116, 1992
Printed in Great Britain.
0021–8928/92 $15.00+.00

SIMPLE WAVES IN PRANDTL–REUSS EQUATIONS†

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(*Received* 20 *April* 1991)

The solution of the system of equations of plane simple waves in a Prandtl–Reuss isotropically work-hardening medium is reduced in general (without any assumptions on the form of the work-hardening function and the state in front of the simple wave) to the investigation of an ordinary differential equation of the first order. In the special case of linear work-hardening, and also without work-hardening, the solution of the system of equations for plane simple waves is obtained in quadratures. The problem of an oblique shock on a prestressed half-space with arbitrary uniform constant stresses is solved for a linearly work-hardening medium.

FOR THE Prandtl–Reuss equations, the corresponding system of ordinary differential equations of plane simple waves sometimes splits (because the component equations are uncoupled) and thus admits of a straightforward analysis. Plane simple waves propagating along the x^1 axis of the

† *Prikl. Mat. Mekh.* Vol. 56, No. 1, pp. 124–133, 1992.